An Introduction to Isabelle/HOL

2009-1

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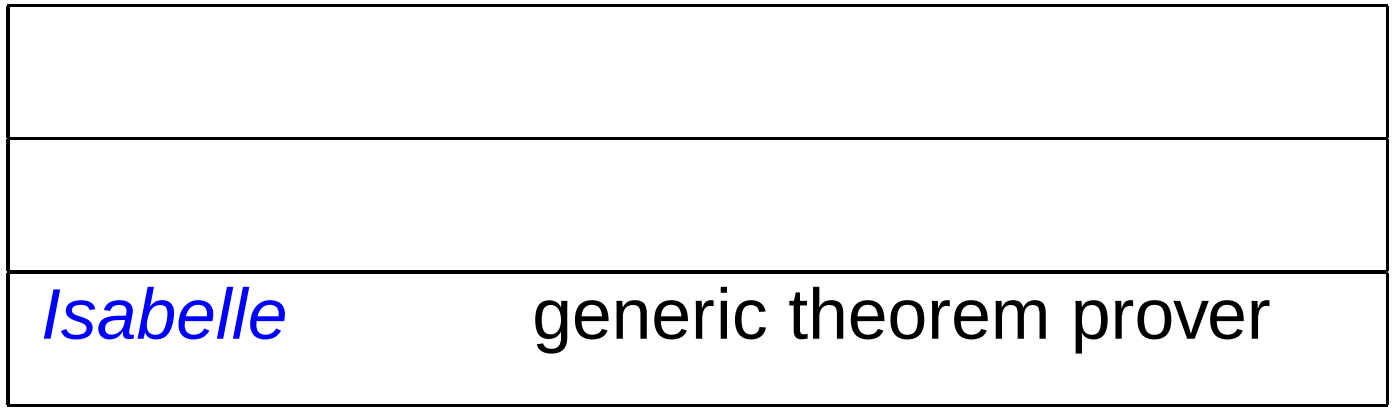
Overview of Isabelle/HOL

System Architecture



System Architecture

Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover

System Architecture

ProofGeneral	(X)Emacs based interface
Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover

HOL

HOL = Higher-Order Logic

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- recursive functions
- logical operators ($\wedge$, $\longrightarrow$, $\forall$, $\exists$, . . .)

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- recursive functions
- logical operators ($\wedge$, $\longrightarrow$, $\forall$, $\exists$, . . .)

HOL is a programming language!

Higher-order = functions are values, too!

Formulae

Syntax (in decreasing priority):

$$
\begin{array}{lcllll}
form & ::= & (form) & \mid & term = term & \mid \quad \neg form \\
 & \mid & form \wedge form & \mid & form \vee form & \mid \quad form \longrightarrow form \\
 & \mid & \forall x.\ form & \mid & \exists x.\ form &
\end{array}
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Scope of quantifiers: as far to the right as possible

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- $\wedge$, $\vee$ and $\longrightarrow$ associate to the right:
  $A \wedge B \wedge C \ \equiv\ A \wedge (B \wedge C)$

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- $A \longrightarrow B \longrightarrow C \equiv A \longrightarrow (B \longrightarrow C) \not\equiv (A \longrightarrow B) \longrightarrow C$ **!**

Warning

Quantifiers have low priority and need to be parenthesized:

! $P \wedge \forall x.\ Q\ x \rightsquigarrow P \wedge (\forall x.\ Q\ x)$!

Types and Terms

Types

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	$\mid$	*'a* $\mid$ *'b* $\mid \ldots$	type variables
	$\mid$	$\tau \Rightarrow \tau$	total functions
	$\mid$	$\tau \times \tau$	pairs (ascii: $*$)
	$\mid$	τ *list*	lists
	$\mid$	$\ldots$	user-defined types

Parentheses: *T1* $\Rightarrow$ *T2* $\Rightarrow$ *T3* $\equiv$ *T1* $\Rightarrow$ *(T2* $\Rightarrow$ *T3)*

Terms: Basic syntax

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Examples: *f (g x) y* *h (λx. f (g x))*

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Examples: $f\ (g\ x)\ y$ $h\ (\lambda x.\ f\ (g\ x))$

Parantheses: $f\ a_1\ a_2\ a_3 \equiv ((f\ a_1)\ a_2)\ a_3$

λ-calculus on one slide

Informal notation: $t[x]$

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Example: $(\lambda\ x.\ x + 5)\ 3 \quad \longrightarrow_\beta \quad (3 + 5)$

$\longrightarrow_\beta$ *in Isabelle: Don't worry, be happy*

Isabelle performs β-reduction automatically

Isabelle considers $(\lambda x.t[x])a$ and $t[a]$ equivalent

Terms and Types

Terms must be well-typed

(the argument of every function call must be of the right type)

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Notation: $t :: \tau$ means t is a well-typed term of type τ.

Type inference

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User can help with type annotations inside the term.

Example: *f (x::nat)*

Currying

Thou shalt curry your functions

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- Curried: *f* :: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: *f'* :: $\tau_1 \times \tau_2 \Rightarrow \tau$

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- Curried: *f* :: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
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Advantage: *partial application* *f* a_1 with a_1 :: τ_1

Terms: Syntactic sugar

Some predefined syntactic sugar:

- *Infix:* +, -, *, #, @, . . .
- *Mixfix:* *if* _ *then* _ *else* _, *case* _ *of*, . . .

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Prefix binds more strongly than infix:

! $f\,x + y \equiv (f\,x) + y \not\equiv f\,(x + y)$!

Terms: Syntactic sugar

Some predefined syntactic sugar:

- *Infix:* +, -, *, #, @, . . .
- *Mixfix: if _ then _ else _*, *case _ of*, . . .

Prefix binds more strongly than infix:

! $f\ x + y \equiv (f\ x) + y \not\equiv f\ (x + y)$!

Enclose *if* and *case* in parentheses:

! *(if _ then _ else _)* !

Base types: bool, nat, list

Type bool

Formulae = terms of type *bool*

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True :: bool
False :: bool
$\wedge, \vee, \ldots$ *:: bool* $\Rightarrow$ *bool* $\Rightarrow$ *bool*
$\vdots$

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if-and-only-if: =

Type nat

0 :: nat
Suc :: nat $\Rightarrow$ *nat*
+, *, ... :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*
$\vdots$

Type nat

0 :: nat
Suc :: nat ⇒ nat
+, *, ... :: *nat ⇒ nat ⇒ nat*
⋮

! Numbers and arithmetic operations are overloaded:
0,1,2,... :: 'a, + :: *'a ⇒ 'a ⇒ 'a*

You need type annotations: *1 :: nat, x + (y::nat)*

Type nat

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Suc :: nat $\Rightarrow$ *nat*
+, *, ... :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*
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0,1,2,... :: 'a, + :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a*

You need type annotations: *1 :: nat, x* + *(y::nat)*

. . . unless the context is unambiguous: *Suc z*

Type list

- *[]*: empty list
- *x # xs*: list with first element *x* ("*head*") and rest *xs* ("*tail*")
- Syntactic sugar: *[x$_1$,...,x$_n$]*

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Large library:
hd, *tl*, *map*, *length*, *filter*, *set*, *nth*, *take*, *drop*, *distinct*, ...

Don't reinvent, reuse!
$\leadsto$ `HOL/List.thy`

Isabelle Theories

Theory = Module

Syntax: `theory` $MyTh$
`imports` $ImpTh_1 \ldots ImpTh_n$
`begin`
(declarations, definitions, theorems, proofs, ...)*
`end`

- $MyTh$: name of theory. Must live in file $MyTh$`.thy`
- $ImpTh_i$: name of *imported* theories. Import transitive.

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Usually: `theory` $MyTh$
`imports Main`
⋮

Proof General

An Isabelle Interface

by David Aspinall

Proof General

Customized version of (x)emacs:

- all of emacs (info: `C-h i`)
- Isabelle aware (when editing `.thy` files)
- mathematical symbols (“x-symbols”)

X-Symbols

Input of funny symbols in Proof General

- via menu (“X-Symbol”)
- via ascii encoding (similar to LaTeX): `\<and>`, `\<or>`, ...
- via abbreviation: `/\`, `\/`, `-->`, ...

x-symbol	∀	∃	λ	¬	∧	∨	⟶	⇒
ascii (1)	`\<forall>`	`\<exists>`	`\<lambda>`	`\<not>`	`/\`	`\/`	`-->`	`=>`
ascii (2)	`ALL`	`EX`	`%`	`~`	`&`	`|`		

(1) is converted to x-symbol, (2) stays ascii.

Demo: terms and types

An introduction to recursion and induction

A recursive datatype: toy lists

datatype *'a list = Nil | Cons 'a ('a list)*

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***Nil*:** empty list

***Cons x xs*:** head *x :: 'a*, tail *xs :: 'a list*

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A toy list: *Cons False (Cons True Nil)*

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***Nil*:** empty list

***Cons x xs*:** head *x :: 'a*, tail *xs :: 'a list*

A toy list: *Cons False (Cons True Nil)*

Predefined lists: *[False, True]*

Structural induction on lists

P xs holds for all lists *xs* if

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- *P Nil*

Structural induction on lists

P xs holds for all lists *xs* if

- *P Nil*
- and for arbitrary *x* and *xs*, *P xs* implies *P (Cons x xs)*

A recursive function: append

Definition by *primitive recursion*:

primrec *app :: 'a list ⇒ 'a list ⇒ 'a list* **where**
app Nil ys = ? |
app (Cons x xs) ys = ??

A recursive function: append

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primrec *app :: 'a list* $\Rightarrow$ *'a list* $\Rightarrow$ *'a list* **where**
app Nil ys = ? |
app (Cons x xs) ys = ??

1 rule per constructor
Recursive calls must drop the constructor $\Longrightarrow$ Termination

Concrete syntax

In `.thy` files:

Types and formulas need to be inclosed in "

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Except for single identifiers, e.g. *'a*

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In `.thy` files:

Types and formulas need to be inclosed in "

Except for single identifiers, e.g. *'a*

" normally not shown on slides

Demo: append and reverse

Proofs

General schema:

lemma $name$`: "..."`
apply `(...)`
apply `(...)`
⋮
done

If the lemma is suitable as a simplification rule:

lemma $name$`[simp]: "..."`

Proof methods

- Structural induction
  - Format: *(induct x)*
    x must be a free variable in the first subgoal.
    The type of *x* must be a datatype.
  - Effect: generates 1 new subgoal per constructor
- Simplification and a bit of logic
  - Format: *auto*
  - Effect: tries to solve as many subgoals as possible using simplification and basic logical reasoning.

Top down proofs

Command

sorry

“completes” any proof.

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Allows top down development:

Assume lemma first, prove it later.

Some useful tools

Disproving tools

Automatic counterexample search by random testing:

quickcheck

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quickcheck

Counterexample search via SAT solver:

nitpick

Finding theorems

1. Click on Find button
2. Input search pattern (e.g. *"_ & True"*)

Demo: Disproving and Finding

Isabelle's meta-logic

Basic constructs

Implication $\Longrightarrow$ (==>)

For separating premises and conclusion of theorems

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Universal quantifier $\bigwedge$ (! !)

For binding local variables

Basic constructs

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Equality $\equiv$ (==)

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Universal quantifier $\bigwedge$ (! !)

For binding local variables

Do not use *inside* HOL formulae

Notation

$$[\![A_1; \ldots ; A_n]\!] \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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$;$ $\approx$ “and”

The proof state

$1.\ \bigwedge x_1 \ldots x_p.\ [\![A_1; \ldots ; A_n]\!] \Longrightarrow B$

$x_1 \ldots x_p$	Local constants
$A_1 \ldots A_n$	Local assumptions
B	Actual (sub)goal

Type and function definition in Isabelle/HOL

Type definition in Isabelle/HOL

Introducing new types

Keywords:

- **typedecl**: pure declaration
- **types**: abbreviation
- **datatype**: recursive datatype

typedecl

typedecl $name$

Introduces new “opaque” type $name$ without definition

typedecl

typedecl $name$

Introduces new “opaque” type $name$ without definition

Example:

typedecl *addr* — An abstract type of addresses

types

types $name = \tau$

Introduces an *abbreviation* $name$ for type τ

types

types $name = \tau$

Introduces an *abbreviation* $name$ for type τ

Examples:

types
name = string
('a,'b)foo = 'a list $\times$ *'b list*

types

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Introduces an *abbreviation* $name$ for type τ

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Type abbreviations are expanded immediately after parsing
Not present in internal representation and Isabelle output

datatype

The example

datatype *'a list = Nil | Cons 'a ('a list)*

Properties:

- Types: *Nil :: 'a list*
  Cons :: 'a ⇒ 'a list ⇒ 'a list
- Distinctness: *Nil ≠ Cons x xs*
- Injectivity: *(Cons x xs = Cons y ys) = (x = y ∧ xs = ys)*

The general case

datatype $(\alpha_1, \ldots, \alpha_n)\tau \quad = \quad C_1\ \tau_{1,1} \ldots \tau_{1,n_1}$

$\qquad\qquad\qquad\qquad\qquad\quad \mid \quad \ldots$

$\qquad\qquad\qquad\qquad\qquad\quad \mid \quad C_k\ \tau_{k,1} \ldots \tau_{k,n_k}$

- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)\tau$
- *Distinctness:* $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- *Injectivity:*
  $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

The general case

datatype $(\alpha_1, \ldots, \alpha_n)\tau$ = $C_1\ \tau_{1,1} \ldots \tau_{1,n_1}$
| $\ldots$
| $C_k\ \tau_{k,1} \ldots \tau_{k,n_k}$

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- *Injectivity:*
  $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

Distinctness and Injectivity are applied automatically
Induction must be applied explicitly

Function definition in Isabelle/HOL

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

Why nontermination can be harmful

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Subtract $f\ x$ on both sides.

$$\Longrightarrow 0 = 1$$

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$$\Longrightarrow 0 = 1$$

! All functions in HOL must be total **!**

Function definition schemas in Isabelle/HOL

- Non-recursive with **definition**
  No problem

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  No problem
- Primitive-recursive with **primrec**
  Terminating by construction
- Well-founded recursion with **fun**
  Automatic termination proof
- Well-founded recursion with **function**
  User-supplied termination proof

definition

Definition (non-recursive) by example

definition *sq :: nat ⇒ nat* **where** *sq n = n * n*

Definitions: pitfalls

definition *prime :: nat* $\Rightarrow$ *bool* **where**
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Unfolding the definition of *sq*:

apply*(unfold sq_def)*

primrec

The example

primrec *app :: 'a list* $\Rightarrow$ *'a list* $\Rightarrow$ *'a list* **where**

app Nil ys = ys |

app (Cons x xs) ys = Cons x (app xs ys)

The general case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \cdots \Rightarrow \tau \Rightarrow \cdots \Rightarrow \tau'$ can be defined by *primitive recursion*:

$$
\begin{array}{lcl}
f\ x_1 \ldots (C_1\ y_{1,1} \ldots y_{1,n_1}) \ldots x_p & = & r_1 \mid \\
\vdots & & \\
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The recursive calls in r_i must be *structurally smaller*, i.e. of the form $f\ a_1 \ldots y_{i,j} \ldots a_p$

nat is a datatype

datatype *nat = 0* | *Suc nat*

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datatype *nat = 0 | Suc nat*

Functions on *nat* definable by primrec!

primrec *f :: nat ⇒ ...*
f 0 = ...
f(Suc n) = ... f n ...

More predefined types and functions

Type option

datatype *'a option = None | Some 'a*

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Important application:

$$\ldots \Rightarrow \textit{'a option} \approx \text{partial function:}$$

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primrec *lookup :: 'k* $\Rightarrow$ *('k* $\times$ *'v) list* $\Rightarrow$ *'v option* **where**

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Example:

primrec *lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option* **where**

lookup k [] = None |

lookup k (x#xs) =

(if fst x = k then Some(snd x) else lookup k xs)

case

Datatype values can be taken apart with *case* expressions:

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Needs *( )* in context

Proof by case distinction

If $t :: \tau$ and τ is a datatype

apply(*case_tac* t)

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If $t :: \tau$ and τ is a datatype

apply(*case_tac* t)

creates k subgoals

$$t = C_i\ x_1 \ldots x_p \Longrightarrow \ldots$$

one for each constructor C_i of type τ.

Demo: trees

fun

From primitive recursion to arbitrary pattern matching

Example: Fibonacchi

fun *fib :: nat* $\Rightarrow$ *nat* **where**

fib 0 = 0 |
fib (Suc 0) = 1 |
fib (Suc(Suc n)) = fib (n+1) + fib n

Example: Separation

fun *sep :: 'a ⇒ 'a list ⇒ 'a list* **where**

sep a [] = [] |
sep a [x] = [x] |
sep a (x#y#zs) = x # a # sep a (y#zs)

Example: Ackermann

fun *ack :: nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat* **where**

```
ack 0         n         = Suc n  |
ack (Suc m) 0           = ack m (Suc 0)  |
ack (Suc m) (Suc n) = ack m (ack (Suc m) n)
```

Key features of fun

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- Termination must be provable
  by lexicographic combination of size measures

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- *size(n::nat) = n*

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- *size(xs) = length xs*

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- *size* counts number of (non-nullary) constructors

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Either the first component decreases, or it stays unchanged and the second component decreases:

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Theorem If each component ordering terminates, then their *lexicographic product* terminates, too.

Ackermann terminates

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Note: order of arguments not important for Isabelle!

Computation Induction

If $f :: \tau \Rightarrow \tau'$ is defined by **fun**, a special induction schema is provided to prove $P(x)$ for all $x :: \tau$:

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Induction follows course of (terminating!) computation

Computation Induction: Example

fun *div2 :: nat* $\Rightarrow$ *nat* **where**
div2 0 = 0 |
div2 (Suc 0) = 0 |
div2(Suc(Suc n)) = Suc(div2 n)

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fun *div2 :: nat* $\Rightarrow$ *nat* **where**
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$\rightsquigarrow$ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \quad P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

Demo: fun

Proof by Simplification

Overview

- Term rewriting foundations
- Term rewriting in Isabelle/HOL
  - Basic simplification
  - Extensions

Term rewriting foundations

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Terminology: equation $\leadsto$ *rewrite rule*

An example

Equations:

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\begin{aligned}
0 + n &= n && (1)\\
(Suc\ m) + n &= Suc\ (m + n) && (2)\\
(Suc\ m \leq Suc\ n) &= (m \leq n) && (3)\\
(0 \leq m) &= True && (4)
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$$[\![P_1 \dots P_n]\!] \Longrightarrow l = r$$

is *applicable* to term $t[s]$ with σ if

- $\sigma(l) = s$ and
- $\sigma(P_1)$, . . . , $\sigma(P_n)$ are provable (again by rewriting).

Interlude: Variables in Isabelle

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Schematic variables:

- Logically: free = schematic
- Operationally:
  - free variables are fixed
  - schematic variables are instantiated by substitutions

From x to ?x

State lemmas with free variables:

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Now usable with arbitrary values for *?xs*

Example: rewriting

rev(a @ []) = rev a

using *app_Nil2* with $\sigma = \{?xs \mapsto a\}$

Term rewriting in Isabelle

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apply(*simp add:* $eq_1 \ldots eq_n$)

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Variations:

- (*simp* ... *del:* ...) removes *simp*-lemmas
- *add* and *del* are optional

auto versus simp

- *auto* acts on all subgoals
- *simp* acts only on subgoal 1
- *auto* applies *simp* and more

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n < *m* $\Longrightarrow$ (*n* < *Suc m*) = *True*
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Rewriting with definitions

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They must be used explicitly: *(simp add: f_def . . .)*

Extensions of rewriting

Local assumptions

Simplification of $A \longrightarrow B$:

1. Simplify A to A'
2. Simplify B using A'

Case splitting with simp

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$$=$$
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By hand: *(simp split: nat.split)*

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Similar for any datatype *t*: *t.split*

Ordered rewriting

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For types *nat*, *int* etc:

- lemmas *add_ac* sort any sum ($+$)
- lemmas *times_ac* sort any product ($*$)

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Problem: $?x + ?y = ?y + ?x$ does not terminate

Solution: permutative *simp*-rules are used only if the term becomes lexicographically smaller.

Example: $b + a \rightsquigarrow a + b$ but not $a + b \rightsquigarrow b + a$.

For types *nat*, *int* etc:

- lemmas *add_ac* sort any sum ($+$)
- lemmas *times_ac* sort any product ($*$)

Example: *(simp add: add_ac)* yields

$$(b + c) + a \rightsquigarrow \cdots \rightsquigarrow a + (b + c)$$

Preprocessing

simp-rules are preprocessed (recursively) for maximal simplification power:

$$\begin{aligned} \neg A &\mapsto A = False \\ A \longrightarrow B &\mapsto A \Longrightarrow B \\ A \wedge B &\mapsto A,\ B \\ \forall x.A(x) &\mapsto A(?x) \\ A &\mapsto A = True \end{aligned}$$

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Example:

$$(p \longrightarrow q \wedge \neg\, r) \wedge s \quad \mapsto \quad \left\{ \begin{array}{r} p \Longrightarrow q = True \\ p \Longrightarrow r = False \\ s = True \end{array} \right\}$$

When everything else fails: Tracing

Set trace mode on/off in Proof General:

Isabelle $\rightarrow$ Settings $\rightarrow$ Trace simplifier

Output in separate `trace` buffer

Demo: simp

Induction heuristics

Basic heuristics

Theorems about recursive functions are proved by induction

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Theorems about recursive functions are proved by induction

Induction on argument number i of f
if f is defined by recursion on argument number i

A tail recursive reverse

primrec *itrev :: 'a list ⇒ 'a list ⇒ 'a list*

A tail recursive reverse

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itrev [] ys = ys |

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Why in this direction?

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itrev [] ys = ys |

itrev (x#xs) ys = itrev xs (x#ys)

lemma *itrev xs [] = rev xs*

Why in this direction?

Because the lhs is “more complex” than the rhs.

Demo

Generalisation

- Replace constants by variables

Generalisation

- Replace constants by variables
- Generalize free variables
  - by $\forall$ in formula
  - by *arbitrary* in induction proof

HOL: Propositional Logic

Overview

- Natural deduction
- Rule application in Isabelle/HOL

Rule notation

$$\frac{A_1 \ldots A_n}{A} \qquad \text{instead of} \qquad [\![A_1 \ldots A_n]\!] \Longrightarrow A$$

Natural Deduction

Natural deduction

Two kinds of rules for each logical operator $\oplus$:

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Introduction: how can I prove $A \oplus B$?

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Two kinds of rules for each logical operator $\oplus$:

Introduction: how can I prove $A \oplus B$?

Elimination: what can I prove from $A \oplus B$?

Natural deduction for propositional logic

$$\frac{}{A \wedge B}\ \texttt{conjI} \qquad \frac{A \wedge B}{\quad}\ \texttt{conjE}$$

$$\frac{A}{A \vee B}\ \frac{B}{A \vee B}\ \texttt{disjI1/2} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ C\end{array} \quad \begin{array}{c}[B]\\ C\end{array}}{C}\ \texttt{disjE}$$

$$\frac{\begin{array}{c}[A]\\ B\end{array}}{A \rightarrow B}\ \texttt{impI} \qquad \frac{A \rightarrow B \quad A}{B}\ \texttt{impE}$$

$$\frac{A \rightarrow B \quad B \rightarrow A}{A \leftrightarrow B}\ \texttt{iffI} \qquad \frac{A \leftrightarrow B}{A \rightarrow B}\ \texttt{iffD1} \qquad \frac{A \leftrightarrow B}{B \rightarrow A}\ \texttt{iffD2}$$

$$\frac{\begin{array}{c}[A]\\ \bot\end{array}}{\neg A}\ \texttt{notI} \qquad \frac{\neg A \quad A}{\bot}\ \texttt{notE}$$

Natural deduction for propositional logic

$$\frac{A \quad B}{A \wedge B}\ \texttt{conjI}$$

$$\frac{A \wedge B}{\quad}\ \texttt{conjE}$$

$$\frac{A}{A \vee B}\ \texttt{disjI1/2} \qquad \frac{B}{A \vee B}\ \texttt{disjI1/2}$$

$$\frac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C}\ \texttt{disjE}$$

$$\frac{\begin{array}{c}[A] \\ B\end{array}}{A \Rightarrow B}\ \texttt{impI}$$

$$\frac{A \Rightarrow B \quad A}{B}\ \texttt{impE}$$

$$\frac{A \Rightarrow B \quad B \Rightarrow A}{A \Leftrightarrow B}\ \texttt{iffI}$$

$$\frac{A \Leftrightarrow B \quad A}{B}\ \texttt{iffD1} \qquad \frac{A \Leftrightarrow B \quad B}{A}\ \texttt{iffD2}$$

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$$\frac{B}{A \Rightarrow B}\ \texttt{impI} \qquad \frac{A \Rightarrow B \quad A}{B}\ \texttt{impE}$$

$$\frac{A \Rightarrow B \quad B \Rightarrow A}{A \Leftrightarrow B}\ \texttt{iffI} \qquad \frac{A \Leftrightarrow B}{A \Rightarrow B}\ \texttt{iffD1} \qquad \frac{A \Leftrightarrow B}{B \Rightarrow A}\ \texttt{iffD2}$$

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$$\frac{}{A \Rightarrow A}\ \texttt{impI} \qquad \frac{A \Rightarrow B \quad A}{B}\ \texttt{impE}$$

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Natural deduction for propositional logic

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Operational reading

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To prove A it suffices to prove $A_1 \ldots A_n$.

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$$\frac{A_1 \ldots A_n}{A}$$

Introduction rule:

To prove A it suffices to prove $A_1 \ldots A_n$.

Elimination rule

If I know A_1 and want to prove A
it suffices to prove $A_2 \ldots A_n$.

Equality

$$\frac{}{t = t}\ \texttt{refl} \qquad \frac{s = t}{t = s}\ \texttt{sym} \qquad \frac{r = s \quad s = t}{r = t}\ \texttt{trans}$$

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$$\frac{s = t \quad A(s)}{A(t)}\ \texttt{subst}$$

Rarely needed explicitly — used implicitly by *simp*

More rules

$$\frac{A \longrightarrow B \quad A}{B}\ \mathtt{mp}$$

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$$\frac{\neg A \Longrightarrow \mathit{False}}{A}\ \texttt{ccontr} \qquad \frac{\neg A \Longrightarrow A}{A}\ \texttt{classical}$$

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Remark:

`ccontr` and `classical` are not derivable from the ND-rules.

More rules

$$\frac{A \longrightarrow B \quad A}{B}\ \texttt{mp}$$

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Remark:

`ccontr` and `classical` are not derivable from the ND-rules.

They make the logic “classical”, i.e. “non-constructive”.

Proof by assumption

$$\frac{A_1 \quad \ldots \quad A_n}{A_i}\ \texttt{assumption}$$

Rule application: the rough idea

Applying rule $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$ to subgoal C:

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Working backwards, like in Prolog!

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Example: rule: $[\![?P; ?Q]\!] \Longrightarrow ?P \wedge ?Q$

subgoal: $1.\ A \wedge B$

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Applying rule $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$ to subgoal C:

- Unify A and C
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Working backwards, like in Prolog!

Example: rule: $[\![?P; ?Q]\!] \Longrightarrow ?P \wedge ?Q$
subgoal: $1.\ A \wedge B$

Result: $1.\ A$
$2.\ B$

Rule application: the details

Rule: $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$

Subgoal: *1.* $[\![B_1; \ldots ; B_m]\!] \Longrightarrow C$

Rule application: the details

Rule: $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$

Subgoal: *1.* $[\![B_1; \ldots ; B_m]\!] \Longrightarrow C$

Substitution: $\sigma(A) \equiv \sigma(C)$

Rule application: the details

Rule: $[\![A_1; \ldots ; A_n]\!] \Longrightarrow A$

Subgoal: $1.\ [\![B_1; \ldots ; B_m]\!] \Longrightarrow C$

Substitution: $\sigma(A) \equiv \sigma(C)$

New subgoals:

$1.\ \sigma([\![B_1; \ldots ; B_m]\!] \Longrightarrow A_1)$

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Command:

apply(*rule <rulename>*)

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apply *assumption*

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by unifying C with one of the B_i

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Like *rule* but also

- unifies first premise of rule with an assumption
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New subgoal:	$1.\ [\![\ X;\ Y\ ]\!] \Longrightarrow [\![\ A;\ B\ ]\!] \Longrightarrow Z$
same as:	$1.\ [\![\ X;\ Y;\ A;\ B\ ]\!] \Longrightarrow Z$

How to prove it by natural deduction

- Intro rules decompose formulae to the right of $\Longrightarrow$.

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apply(*rule <intro-rule>*)

- Elim rules decompose formulae on the left of $\Longrightarrow$.

apply(*erule <elim-rule>*)

Demo: propositional proofs

$\Longrightarrow$ *vs* $\longrightarrow$

To facilitate application of theorems:

write them like this $[\![A_1; \ldots; A_n]\!] \Longrightarrow A$

not like this $A_1 \wedge \ldots \wedge A_n \longrightarrow A$

HOL: Predicate Logic

Parameters

Subgoal:

1. $\bigwedge x_1 \ldots x_n$*. Formula*

The x_i are called parameters of the subgoal.
Intuition: local constants, i.e. arbitrary but fixed values.

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Intuition: local constants, i.e. arbitrary but fixed values.

Rules are automatically lifted over $\bigwedge x_1 \ldots x_n$ and applied directly to *Formula*.

Scope

- Scope of parameters: whole subgoal
- Scope of $\forall$, $\exists$, . . . : ends with *;* or $\Longrightarrow$

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- Scope of parameters: whole subgoal
- Scope of $\forall$, $\exists$, ...: ends with *;* or $\Longrightarrow$

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\ \ Q\ x\ y\ ]\!] \Longrightarrow \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ [\![\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\ \ Q\ x\ y\ ]\!] \Longrightarrow \exists x_1.\ Q\ x_1\ y$$

α-Conversion

Bound variables are renamed automatically to avoid name clashes with other variables.

Natural deduction for quantifiers

$$\frac{}{\forall x.\ P(x)}\ \texttt{allI} \qquad \frac{\qquad}{\qquad}\ \texttt{allE}$$

$$\frac{}{\qquad}\ \texttt{exI} \qquad \frac{\qquad}{\qquad}\ \texttt{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P(x)}{\forall x.\ P(x)}\ \texttt{allI} \qquad \frac{\forall x.\ P(x)}{P(t)}\ \texttt{allE}$$

$$\frac{P(t)}{\exists x.\ P(x)}\ \texttt{exI} \qquad \frac{\exists x.\ P(x) \qquad \bigwedge x.\ P(x) \Longrightarrow Q}{Q}\ \texttt{exE}$$

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$$\frac{\bigwedge x.\ P(x)}{\forall x.\ P(x)}\ \texttt{allI} \qquad \frac{\forall x.\ P(x)}{\bigwedge x.\ P(x)}\ \texttt{allE}$$

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Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P(x)}{\forall x.\ P(x)}\ \texttt{allI} \qquad\qquad \texttt{allE}\ \frac{\forall x.\ P(x)}{P(?x)}$$

$$\frac{P(?x)}{\exists x.\ P(x)}\ \texttt{exI} \qquad\qquad \texttt{exE}\ \frac{\exists x.\ P(x)}{P(x)}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P(x)}{\forall x.\ P(x)}\ \texttt{allI} \qquad \frac{\forall x.\ P(x)}{R}\ \texttt{allE}$$

$$\frac{P(?x)}{\exists x.\ P(x)}\ \texttt{exI} \qquad \frac{}{}\ \texttt{exE}$$

Natural deduction for quantifiers

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- `allI` and `exE` introduce new parameters ($\bigwedge x$).

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$$\frac{P(?x)}{\exists x.\ P(x)}\ \texttt{exI} \qquad \frac{\exists x.\ P(x) \quad \bigwedge x.\ P(x) \Longrightarrow R}{R}\ \texttt{exE}$$

- `allI` and `exE` introduce new parameters ($\bigwedge x$).
- `allE` and `exI` introduce new unknowns ($?x$).

Instantiating rules

apply(*rule_tac x =* $term$ *in* $rule$)

Like *rule*, but *?x* in $rule$ is instantiated by $term$ before application.

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! *x* is in $rule$, not in the goal **!**

A quantifier proof

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1. ⋀a. a = a

apply*(rule refl)*

Demo: quantifier proofs

More proof methods

Forward proofs: frule and drule

"Forward" rule: $A_1 \Longrightarrow A$

Subgoal: *1.* $[\![B_1; \ldots ; B_n]\!] \Longrightarrow C$

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Command:

apply(*frule* $rulename$)

Like *frule* but also deletes B_i:

apply(*drule* $rulename$)

frule and drule: the general case

Rule: $[\![A_1; \ldots ; A_m]\!] \Longrightarrow A$

Creates additional subgoals:

$$
\begin{aligned}
&1.\ \sigma([\![B_1; \ldots ; B_n]\!] \Longrightarrow A_2) \\
&\vdots \\
&m\text{-}1.\ \sigma([\![B_1; \ldots ; B_n]\!] \Longrightarrow A_m) \\
&m.\ \sigma([\![B_1; \ldots ; B_n; A]\!] \Longrightarrow C)
\end{aligned}
$$

Forward proofs: OF

r[OF $r_1 \ldots r_n$*]*

Prove assumption 1 of theorem *r* with theorem r_1, and assumption 2 with theorem r_2, and . . .

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Rule *r*	$[\![A_1; \ldots ; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots ; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$

r[OF r_1]

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Substitution	$\sigma(B) \equiv \sigma(A_1)$
r[OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

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  Combination of *clarify* and *simp*.

Demo: proof methods

Sets

Overview

- Set notation
- Inductively defined sets

Set notation

Sets

Sets over type *'a*:

'a set

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$$\textit{'a set} \quad = \quad \textit{'a} \Rightarrow \textit{bool}$$

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- . . . (see Tutorial)

Demo: proofs about sets

Bounded quantifiers

- $\forall x \in A.\ P\ x$

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- `ballI`: $(\bigwedge x.\ x \in A \Longrightarrow P\ x) \Longrightarrow \forall x{\in}A.\ P\ x$
- `bspec`: $[\![\forall x{\in}A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$

Bounded quantifiers

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- `bspec`: $[\![\forall x{\in}A.\ P\ x; x \in A]\!] \Longrightarrow P\ x$
- `bexI`: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x{\in}A.\ P\ x$
- `bexE`: $[\![\exists x{\in}A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

Inductively defined sets

Example: even numbers

Informally:

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0 ∈ Ev |

n ∈ Ev ⟹ n + 2 ∈ Ev

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inductive_set $S :: \tau$ *set*

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Format of inductive definitions

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$$\vdots$$

where $A_1; \ldots; A_k$ are side conditions not involving S.

Proving properties of even numbers

Easy: $4 \in Ev$

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- rule $0 \in Ev$
  $\Longrightarrow m = 0 \Longrightarrow 0+0 \in Ev$
- rule $n \in Ev \Longrightarrow n+2 \in Ev$
  $\Longrightarrow m = n+2$ and $n+n \in Ev$ (ind. hyp.!)
  $\Longrightarrow m+m = (n+2)+(n+2) = ((n+n)+2)+2 \in Ev$

Rule induction for Ev

To prove

$$n \in Ev \Longrightarrow P\ n$$

by *rule induction* on $n \in Ev$ we must prove

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by *rule induction* on $n \in Ev$ we must prove

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Rule `Ev.induct`:

$$\llbracket\, n \in Ev;\ P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P(n+2) \,\rrbracket \Longrightarrow P\ n$$

Rule induction in general

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that P is preserved:

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by *rule induction* on $x \in S$
we must prove for every rule

$$[\![\ a_1 \in S; \ldots ; a_n \in S\ ]\!] \Longrightarrow a \in S$$

that P is preserved:

$$[\![\ P\ a_1; \ldots ; P\ a_n\ ]\!] \Longrightarrow P\ a$$

Rule induction in general

Set S is defined inductively.
To prove

$$x \in S \Longrightarrow P\ x$$

by *rule induction* on $x \in S$
we must prove for every rule

$$[\![\ a_1 \in S; \ldots ; a_n \in S\ ]\!] \Longrightarrow a \in S$$

that P is preserved:

$$[\![\ P\ a_1; \ldots ; P\ a_n\ ]\!] \Longrightarrow P\ a$$

In Isabelle/HOL:

apply*(induct rule: S.induct)*

Demo: inductively defined sets

Inductive predicates

$$x \in S \quad \rightsquigarrow \quad S\,x$$

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Example:

inductive *Ev :: nat* $\Rightarrow$ *bool*

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Ev 0 |

Ev n $\Longrightarrow$ *Ev (n + 2)*

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predicate: simpler syntax

set: direct usage of $\cup$ etc

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Comparison:

predicate: simpler syntax

set: direct usage of $\cup$ etc

Inductive predicates can be of type $\tau_1 \Rightarrow ... \Rightarrow \tau_n \Rightarrow$ *bool*

Automating it

simp and auto

simp rewriting and a bit of arithmetic

auto rewriting and a bit of arithmetic, logic & sets

simp and auto

simp rewriting and a bit of arithmetic

auto rewriting and a bit of arithmetic, logic & sets

- Show you where they got stuck

simp and auto

simp rewriting and a bit of arithmetic

auto rewriting and a bit of arithmetic, logic & sets

- Show you where they got stuck
- highly incomplete wrt logic

blast

- A complete (for FOL) tableaux calculus implementation

blast

- A complete (for FOL) tableaux calculus implementation
- Covers logic, sets, relations, . . .

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- Extensible with intro/elim rules

blast

- A complete (for FOL) tableaux calculus implementation
- Covers logic, sets, relations, . . .
- Extensible with intro/elim rules
- Almost no “=”

Demo: blast

blast: A 2-stage implementation

1. Search for proof with quick, dirty and possibly buggy program

blast: A 2-stage implementation

1. Search for proof with quick, dirty and possibly buggy program (while recording proof tree)

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2. Check proof tree with Isabelle kernel

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- Proofs only constructable via small trustworthy kernel

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- Proofs only constructable via small trustworthy kernel
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- Proofs only constructable via small trustworthy kernel
- Proof search programmable in ML on top

Isabelle follows the LCF architecture

fast and friends

fast slow and incomplete version of *blast*

fast and friends

fast slow and incomplete version of *blast*

fastsimp rewriting and logic

fast and friends

fast slow and incomplete version of *blast*

fastsimp rewriting and logic

force slower but completer version of *fastsimp*

metis

- An fast resolution theorem prover in ML

metis

- An fast resolution theorem prover in ML
- Can deal with bidirectional “=”

metis

- An fast resolution theorem prover in ML
- Can deal with bidirectional “=”
- Knows only pure logic, not sets etc

Demo: beyond blast

Problems

- Most proofs require additional lemmas.

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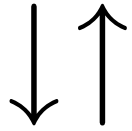
- Adding arbitrary lemmas slows *blast* down significantly; *metis* copes better.
- Finding the right lemmas in a library of thousands of lemmas is light years beyond *blast* and *metis*.
- There are highly optimized *ATPs* (automatic theorem provers) for FOL that can deal with large libraries . . .

Sledgehammer

Architecture

Isabelle

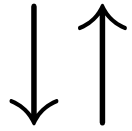


ATPs

Architecture

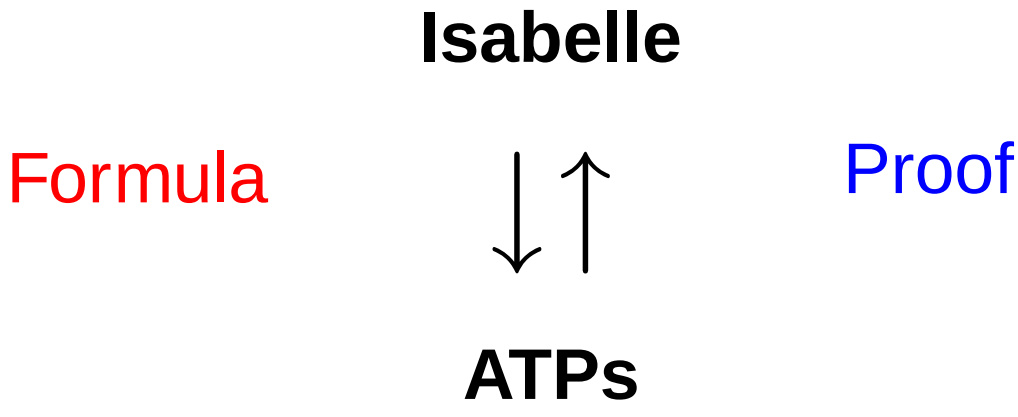
Isabelle

Formula

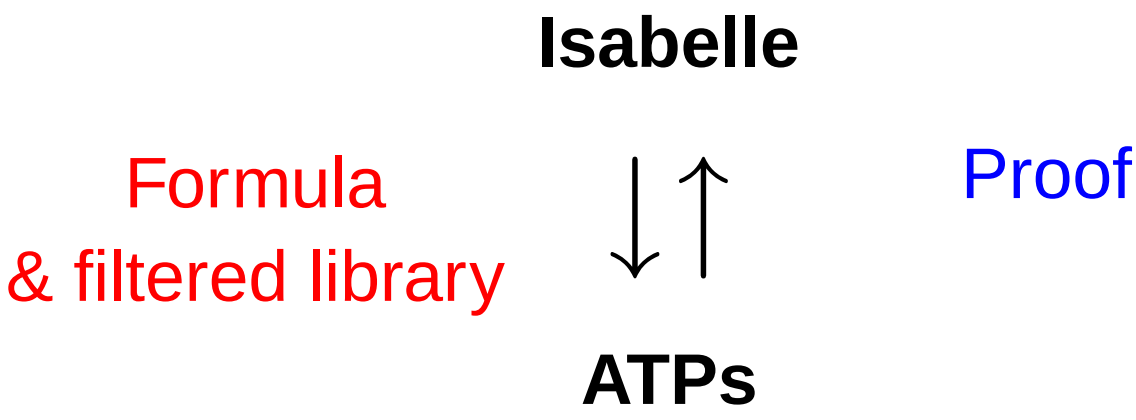


ATPs

Architecture



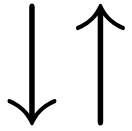
Architecture



Architecture

Isabelle

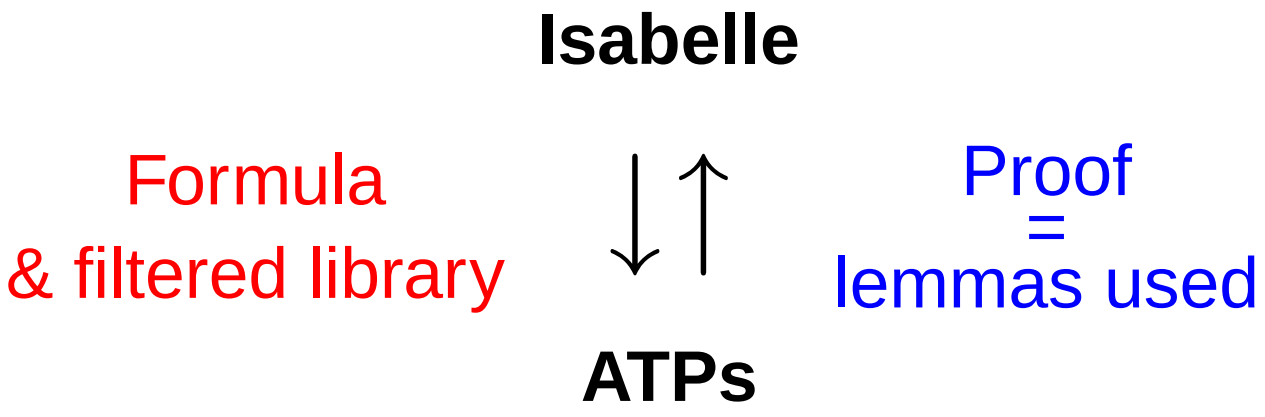
Formula
& filtered library



Proof
=
lemmas used

ATPs

Architecture



Empirical study:

Sledgehammer works for 1/3 of non-trivial Isabelle proofs

Demo: Sledghehammer

Automating Arithmetic

Automating arithmetic

arith:

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- proves linear formulas (no “$*$”)

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- complete for quantifier-free *real* arithmetic

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- complete for first-order theory of *nat* and *int* (Presburger arithmetic)

Automating arithmetic

Theorem (Tarski) $Th(\mathbb{R}, +, -, *, <, =)$ is decidable.

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Automating arithmetic

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An incomplete but (often) fast method for the quantifier-free fragement:

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Idea: (re)write polynomials as sums-of-squares to prove non-negativity

Demo: Arithmetic

Isar — A language for structured proofs

Apply scripts

- unreadable

Apply scripts

- unreadable
- hard to maintain

Apply scripts

- unreadable
- hard to maintain
- do not scale

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No structure!

Apply scripts versus Isar proofs

Apply script = assembly language program

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with comments

Apply scripts versus Isar proofs

Apply script = assembly language program

Isar proof = structured program with comments

But: **apply** still useful for proof exploration

A typical Isar proof

proof

assume $formula_0$

have $formula_1$ **by** *simp*

$\vdots$

have $formula_n$ **by** *blast*

show $formula_{n+1}$ **by** $\ldots$

qed

A typical Isar proof

proof

  assume $formula_0$

  have $formula_1$ **by** *simp*

  $\vdots$

  have $formula_n$ **by** *blast*

  show $formula_{n+1}$ **by** $\ldots$

qed

proves $formula_0 \Longrightarrow formula_{n+1}$

Overview

- Basic Isar
- Isar by example
- Proof patterns
- Streamlining proofs

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = *(simp . . .)* | *(blast . . .)* | *(rule . . .)* | **. . .**

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = *(simp ...)* | *(blast ...)* | *(rule ...)* | **...**

statement = **fix** variables ($\bigwedge$)
| **assume** prop ($\Longrightarrow$)
| [**from** fact$^+$] (**have** | **show**) prop proof

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Isar core syntax

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prop = [name:] "formula"

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = *(simp . . .)* | *(blast . . .)* | *(rule . . .)* | . . .

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prop = [name:] "formula"

fact = name | name**[**OF fact$^+$**]** | ‘formula‘

Isar by example

Example: Cantor's theorem

lemma *Cantor:* $\neg$ *surj(f :: 'a* $\Rightarrow$ *'a set)*

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by(*simp add: surj_def*)

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Example: Cantor's theorem

lemma *Cantor:* $\neg$ *surj(f :: 'a* $\Rightarrow$ *'a set)*
proof assume *surj*, show *False*
  assume *a: surj f*
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    by *blast*
  from *c* **show** *False*
    by *blast*
qed

Demo: this, then etc

Abbreviations

this	=	the previous proposition proved or assumed
then	=	**from** *this*
thus	=	**then show**
hence	=	**then have**

using

First the what, then the how:

(**have**|**show**) prop **using** facts

using

First the what, then the how:

(**have**|**show**) prop **using** facts

=

from facts (**have**|**show**) prop

Example: Structured lemma statement

lemma *Cantor':*
  fixes *f :: 'a ⇒ 'a set*
  assumes *s: surj f*
  shows *False*

Example: Structured lemma statement

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proof -

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qed

Proves $surj\ f \Longrightarrow False$

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proof - no automatic proof step
  have ∃ *a. {x. x ∉ f x} = f a* **using** *s*
    by*(auto simp: surj_def)*
  thus *False* **by** *blast*
qed

Proves $surj\ f \Longrightarrow False$
but *surj f* becomes local fact *s* in proof.

The essence of structured proofs

Assumptions and intermediate facts
can be named and referred to explicitly and selectively

Structured lemma statements

fixes $x :: \tau_1$ **and** $y :: \tau_2$ **. . .**
assumes a: P **and** b: Q **. . .**
shows R

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assumes a: *P* **and** b: *Q* **. . .**
shows *R*

- **fixes** and **assumes** sections optional
- **shows** optional if no **fixes** and **assumes**

Proof patterns

Propositional proof patterns

show $P \longleftrightarrow Q$
proof
  assume P
  $\vdots$
  show Q ...
next
  assume Q
  $\vdots$
  show P ...
qed

Propositional proof patterns

show $P \longleftrightarrow Q$
proof
  assume P
  $\vdots$
  show Q $\ldots$
next
  assume Q
  $\vdots$
  show P $\ldots$
qed

show $A = B$
proof
  show $A \subseteq B$ $\ldots$
next
  show $B \subseteq A$ $\ldots$
qed

Propositional proof patterns

show $P \longleftrightarrow Q$
proof
  assume P
  ⋮
  show Q ...
next
  assume Q
  ⋮
  show P ...
qed

show $A = B$
proof
  show $A \subseteq B$ ...
next
  show $B \subseteq A$ ...
qed

show $A \subseteq B$
proof
  fix x
  assume $x \in A$
  ⋮
  show $x \in B$ ...
qed

Propositional proof patterns

show R
proof *cases*
  assume P
  ⋮
  show R . . .
next
  assume $\neg P$
  ⋮
  show R . . .
qed

Case distinction

Propositional proof patterns

show R
proof *cases*
  assume P
  ⋮
  show R ...
next
  assume $\neg P$
  ⋮
  show R ...
qed

Case distinction

have $P \vee Q$ ...
then show R
proof
  assume P
  ⋮
  show R ...
next
  assume Q
  ⋮
  show R ...
qed

Case distinction

Propositional proof patterns

show R
proof *cases*
  assume P
  ⋮
  show R . . .
next
  assume $\neg P$
  ⋮
  show R . . .
qed

Case distinction

have $P \vee Q$ . . .
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proof
  assume P
  ⋮
  show R . . .
next
  assume Q
  ⋮
  show R . . .
qed

Case distinction

show P
proof *(rule ccontr)*
  assume $\neg P$
  ⋮
  show *False* . . .
qed

Contradiction

Quantifier introduction proof patterns

show $\forall x.\ P(x)$
proof
  fix x *local fixed variable*
  show $P(x)$ . . .
qed

Quantifier introduction proof patterns

show $\forall x.\ P(x)$
proof
  fix *x* *local fixed variable*
  show $P(x)$ . . .
qed

show $\exists x.\ P(x)$
proof
  ⋮
  show $P(witness)$ . . .
qed

∃ *elimination:* obtain

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have $\exists x.\ P(x)$
then obtain x **where** p: $P(x)$ **by** blast
⋮ x local fixed variable

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Works for one or more x

obtain *example*

lemma *Cantor'':* $\neg$ *surj(f :: 'a* $\Rightarrow$ *'a set)*
proof
  assume *surj f*
  hence $\exists$ *a. {x. x* $\notin$ *f x} = f a* **by**(*auto simp: surj_def*)

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  then obtain *a* **where** *{x. x $\notin$ f x} = f a* **by** *blast*

obtain *example*

lemma *Cantor'':* $\neg$ *surj(f :: 'a $\Rightarrow$ 'a set)*
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  assume *surj f*
  hence $\exists$ *a. {x. x $\notin$ f x} = f a* **by**(*auto simp: surj_def*)
  then obtain *a* **where** *{x. x $\notin$ f x} = f a* **by** *blast*
  hence *a $\notin$ f a $\longleftrightarrow$ a $\in$ f a* **by** *blast*

obtain *example*

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  then obtain *a* **where** *{x. x $\notin$ f x} = f a* **by** *blast*
  hence *a $\notin$ f a $\longleftrightarrow$ a $\in$ f a* **by** *blast*
  thus *False* **by** *blast*
qed

proof method

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Applies method and generates subgoal(s):

1. $\bigwedge x_1 \ldots x_n \llbracket A_1; \ldots ; A_m \rrbracket \Longrightarrow A$

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How to prove each subgoal:

fix $x_1 \ldots x_n$
assume $A_1 \ldots A_m$
$\vdots$
show A

proof method

Applies method and generates subgoal(s):

1. $\bigwedge x_1 \ldots x_n \llbracket A_1; \ldots ; A_m \rrbracket \Longrightarrow A$

How to prove each subgoal:

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assume $A_1 \ldots A_m$
$\vdots$
show A

Separated by **next**

Demo: proof

Streamlining proofs: Pattern matching and Quotations

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** *?L* $\longleftrightarrow$ *?R*)

Example: pattern matching

show $formula_1 \longleftrightarrow formula_2$ (**is** *?L* $\longleftrightarrow$ *?R*)
proof
  assume *?L*
  ⋮
  show *?R* . . .
next
  assume *?R*
  ⋮
  show *?L* . . .
qed

?thesis

show $formula$
proof -
  ⋮
  show *?thesis* . . .
qed

?thesis

show $formula$ (is *?thesis*)
proof -
  ⋮
  show *?thesis* . . .
qed

?thesis

show $formula$ (is *?thesis*)
proof -
  ⋮
  show *?thesis* . . .
qed

Every show implicitly defines *?thesis*

Quoting facts by value

By name:

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⋮

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⋮

from x0 . . .

By value:

have "*x* > *0*" . . .

⋮

from ‘*x*>*0*‘ . . .

back quotes

Demo: pattern matching and quotations

Advanced Isar

Overview

- Case distinction
- Induction
- Chains of (in)equations

Case distinction

Demo: case distinction

Datatype case distinction

datatype $t = C_1 \vec{\tau} \mid \ldots$

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datatype $t = C_1\ \vec{\tau} \mid \ldots$

proof *(cases* $term$*)*
  case *(*$C_1\ \vec{x}$*)*
  $\ldots\ \vec{x}\ \ldots$
next
⋮
qed

Datatype case distinction

datatype $t = C_1\ \vec{\tau} \mid \ldots$

proof (*cases* $term$)
  case ($C_1\ \vec{x}$)
  $\ldots\ \vec{x}\ \ldots$
next
$\vdots$
qed

where **case** ($C_i\ \vec{x}$) $\equiv$

fix $\vec{x}$
assume $\underbrace{C_i\text{:}}_{\text{label}}$ $\underbrace{term = (C_i\ \vec{x})}_{\text{formula}}$

Induction

Overview

- Structural induction
- Rule induction
- Induction with fun

Structural induction for type nat

show *P(n)*
proof *(induct n)*
  case *0*
  . . .
  show *?case*
next
  case *(Suc n)*
  . . .
  . . . *n* . . .
  show *?case*
qed

Structural induction for type nat

show *P(n)*
proof *(induct n)*
  case *0* $\equiv$ **let** *?case* = $P(0)$
  . . .
  show *?case*
next
  case *(Suc n)*
  . . .
  . . . *n* . . .
  show *?case*
qed

Structural induction for type nat

show *P(n)*
proof *(induct n)*
  case *0* $\equiv$ **let** *?case* = $P(0)$
  . . .
  show *?case*
next
  case *(Suc n)* $\equiv$ **fix** *n* **assume** *Suc:* $P(n)$
  . . . **let** *?case* = $P(Suc\ n)$
  . . . *n* . . .
  show *?case*
qed

Demo: structural induction

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof *(induct n)*
  case *0*
  . . .
  show *?case*
next
  case *(Suc n)*
  . . .
  . . . n . . .
  . . .
  show *?case*
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof *(induct n)*
  case *0* $\equiv$ **fix** *x* **assume** *0: A(0)*
  . . . **let** *?case = P(0)*
  show *?case*
next
  case *(Suc n)*
  . . .
  . . . *n* . . .
  . . .
  show *?case*
qed

Structural induction with $\Longrightarrow$

show $A(n) \Longrightarrow P(n)$
proof *(induct n)*
  case *0* $\equiv$ **fix** *x* **assume** *0: A(0)*
  . . . **let** *?case = P(0)*
  show *?case*
next
  case *(Suc n)* $\equiv$ **fix** *n*
  . . . **assume** *Suc: A(n)* $\Longrightarrow$ *P(n)*
  . . . *n* . . . *A(Suc n)*
  . . . **let** *?case = P(Suc n)*
  show *?case*
qed

A remark on style

- **case** *(Suc n)* . . . **show** *?case*
  is easy to write and maintain

A remark on style

- **case** *(Suc n)* . . . **show** *?case*
  is easy to write and maintain
- **fix** *n* **assume** $formula$ . . . **show** $formula'$
  is easier to read:
  - all information is shown locally
  - no contextual references (e.g. *?case*)

Demo: structural induction with $\Longrightarrow$

Rule induction

Inductive definition

inductive_set S

intros

$rule_1$: $[\![\, s \in S; A \,]\!] \Longrightarrow s' \in S$

$\vdots$

$rule_n$: $\ldots$

Rule induction

show $x \in S \Longrightarrow P(x)$
proof *(induct rule: S.induct)*
  case $rule_1$
  . . .
  show *?case*
next
⋮
next
  case $rule_n$
  . . .
  show *?case*
qed

Implicit selection of induction rule

assume A: $x \in S$

⋮

show $P(x)$

using A **proof** *induct*

⋮

qed

Implicit selection of induction rule

assume *A:* $x \in S$

⋮

show *P(x)*

using *A* **proof** *induct*

⋮

qed

lemma assumes *A:* $x \in S$ **shows** *P(x)*

using *A* **proof** *induct*

⋮

qed

Renaming free variables in rule

case *(rule*$_i$ *x*$_1$ ... *x*$_k$*)*

Renames the (alphabetically!) first k variables in *rule*$_i$ to *x*$_1$ ... *x*$_k$.

Demo: rule induction

Induction with fun

Definition:

fun *f*

⋮

Induction with fun

Definition:

fun *f*

⋮

Proof:

show . . . *f*(. . .) . . .

proof *(induct* x_1 . . . x_k *rule: f.induct)*

Induction with fun

Definition:

fun *f*

⋮

Proof:

show . . . *f*(. . .) . . .

proof *(induct* x_1 . . . x_k *rule: f.induct)*

  case *1*

  ⋮

Induction with fun

Definition:

fun *f*

⋮

Proof:

show . . . *f*(. . .) . . .

proof *(induct* x_1 . . . x_k *rule: f.induct)*

  case *1*

  ⋮

Case i refers to equation i in the definition of *f*

Induction with fun

Definition:

fun *f*

⋮

Proof:

show . . . *f*(. . .) . . .

proof *(induct* x_1 . . . x_k *rule: f.induct)*

  case *1*

  ⋮

Case i refers to equation i in the definition of *f*
More precisely: to equation i in *f.simps*

Demo: induction with fun

Chains of (in)equations

also

have $"t_0 = t_1"$ . . .

also

have "$t_0 = t_1$" . . .

also

have "$\ldots = t_2$" . . .

also

have "$t_0 = t_1$" ...

also

have "... = t_2" ... ... $\equiv t_1$

also

have "$t_0 = t_1$" ...

also

have "$\ldots = t_2$" ... $\ldots \equiv t_1$

also

⋮

also

have "$\ldots = t_n$" ...

also

have "$t_0 = t_1$" ...

also

have "... $= t_2$" ... ... $\equiv t_1$

also

⋮

also

have "... $= t_n$" ... ... $\equiv t_{n-1}$

also

have "$t_0 = t_1$" ...
also
have "... = t_2" ... ... $\equiv t_1$
also
⋮
also
have "... = t_n" ... ... $\equiv t_{n-1}$
finally show ...

also

have "$t_0 = t_1$" ...
also
have "... = t_2" ... ... $\equiv t_1$
also
⋮
also
have "... = t_n" ... ... $\equiv t_{n-1}$
finally show ...
— like **from** ‘$t_0 = t_n$‘ **show**

also

- “. . .” is merely an abbreviation

also

- “. . .” is merely an abbreviation
- **also** works for other transitive relations ($<$, $\leq$, . . .)

Demo: also

Accumulating facts

moreover

have $formula_1$ **. . .**

moreover

have $formula_1$ **. . .**
moreover
have $formula_2$ **. . .**

moreover

have $formula_1$ ...
moreover
have $formula_2$ ...
moreover
⋮
moreover
have $formula_n$ ...

moreover

have $formula_1$ **. . .**
moreover
have $formula_2$ **. . .**
moreover
⋮
moreover
have $formula_n$ **. . .**
ultimately show . . .

moreover

have $formula_1$ **. . .**
moreover
have $formula_2$ **. . .**
moreover
⋮
moreover
have $formula_n$ **. . .**
ultimately show . . .

— like **from** $f_1 \ldots f_n$ **show** but needs no labels

Demo: moreover