Integrator Analysis

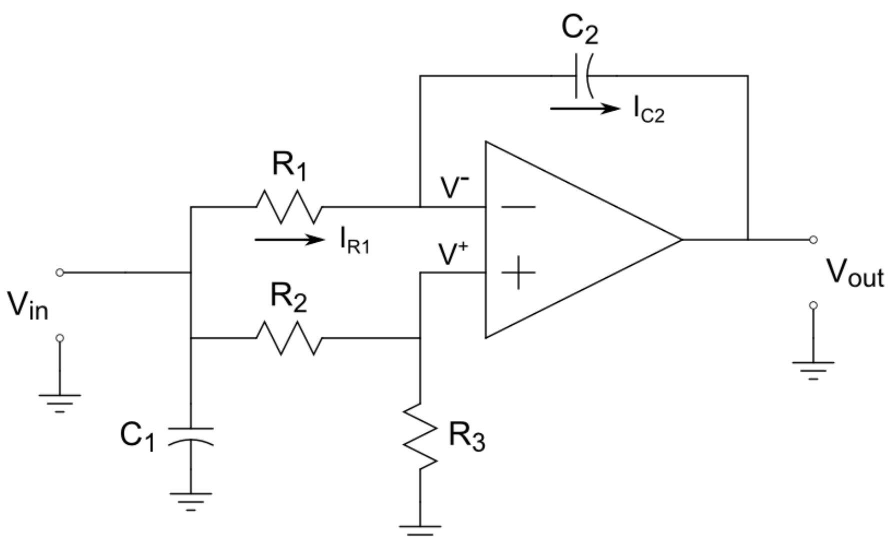
Adam Gulyas

March 2016

This is a steady state algebraic analysis of the integrator sub-block from the 2016 Hackathon Synth VCO. Steady state means I'm assuming that the component values and inputs don't change. This is a pretty good assumption since V_{in} usually changes minimally during one period of the oscillator.

The full circuit contains a switch which adds or removes resistor R_4 from the integrator. The following two cases examine both these situations.

Case 1



SW1 open (Vout falling)

The first thing to note is that C_1 acts to filter V_{in}. It can be ignored if we assume V_{in} is constant over the period of analysis.

Here the switch is open and R_4 is removed from the circuit.

The inputs of an opamp consume no current:

$$i_{R1} = i_{C1}$$

Apply the voltage-current relationships:

$$\frac{V_{in} - V^{-}}{R_1} = C_2 \frac{dV_{c2}}{\mathrm{dt}}$$

$$\frac{V_{in} - V^-}{R_1} = C_2 \frac{\mathrm{d}(V^- - V_{out})}{\mathrm{dt}}$$

The opamp acts to keep $V^- = V^+$:

$$\frac{V_{in} - V^+}{R_1} = C_2 \frac{\mathrm{d}(V^+ - V_{out})}{\mathrm{dt}}$$

R_2 and R_3 are equal and thus act as a 1/2 voltage divider, which outputs half of V_{in} into V^+:

$$\frac{V_{in} - \frac{V_{in}}{2}}{R_1} = C_2 \frac{\mathrm{d}\left(\frac{V_{in}}{2} - V_{out}\right)}{\mathrm{dt}}$$

$$\frac{V_{in}}{2R_1C_2} = \frac{\mathrm{d}\left(\frac{V_{in}}{2} - V_{out}\right)}{\mathrm{dt}}$$

$$\frac{V_{in}}{2R_1C_2} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{V_{in}}{2}\right) - \frac{\mathrm{d}}{\mathrm{dt}}(V_{out})$$

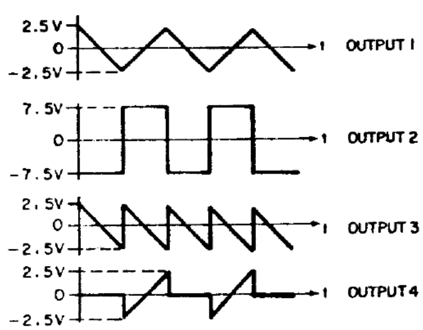
Assume V_{in} is constant over the period of analysis:

$$\frac{V_{in}}{2R_1C_2} = 0 - \frac{\mathrm{d}}{\mathrm{dt}}(V_{out})$$

$$\frac{\mathrm{d}}{\mathrm{dt}}(V_{out}) = \frac{-V_{in}}{2R_1C_2}$$

$$\boxed{V_{out} = \frac{-1}{2R_1C_2} \int V_{in}\, \mathrm{dt}}$$

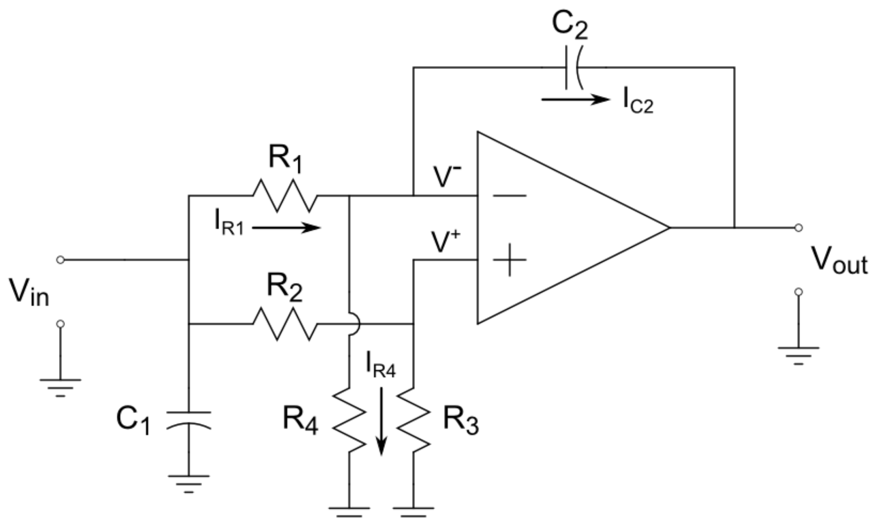
This is the operating equation of the circuit when the switch is open. Note the negative sign. Since all the variables are positive, the integral will ensure V_{out} steadily decreases. This agrees with the outputs shown in the schematic. It can be seen that when the switch is open (output 2 is low), V_{out} (output 1) is falling:



Case 2

In order for this to be an effective oscillator, the operating equation for case 1 should be the negative of case 2. If there are any other terms is will mean there's an unintended extra gain factor for half the period and the triangle wave will be skewed. (Which would make the square wave not be 50% duty cycle, ect.)

Here, the switch is closed and R_4 is included in the circuit:



SW1 closed (Vout rising)

The inputs of an opamp consume no current:

$$i_{R1} = i_{R4} + i_{C1}$$

Apply the voltage-current relationships:

$$\frac{V_{in} - V^-}{R_1} = \frac{V^- - 0}{R_4} + C_2 \frac{\mathrm{d}V_{c2}}{\mathrm{dt}}$$

$$\frac{V_{in} - V^-}{R_1} = \frac{V^-}{R_4} + C_2 \frac{\mathrm{d}(V^- - V_{out})}{\mathrm{dt}}$$

Skipping some steps that were included above:

$$\frac{V_{in}}{2R_1} = \frac{V_{in}}{2R_4} - C_2 \frac{\mathrm{d}(V_{out})}{\mathrm{dt}}$$

$$\frac{-V_{in}}{2C_2}\left(\frac{1}{R_1} - \frac{1}{R_4}\right) = \frac{\mathrm{d}(V_{out})}{\mathrm{dt}}$$

If $R_4 < R_1$ (this restriction is better defined later):

$$\frac{\mathrm{d}(V_{out})}{\mathrm{dt}} = \frac{V_{in}}{2C_2}\left(\frac{1}{R_4} - \frac{1}{R_1}\right)$$

$$\boxed{V_{out} = \frac{1}{2C_2}\left(\frac{1}{R_4} - \frac{1}{R_1}\right)\int V_{in}\,\mathrm{dt}}$$

The gain factor is positive (ensured by the above restriction), meaning V_{out} is increasing. However, the desired fundamental equation relationship, stated algebraically, is $V_{out}|_{\mathrm{SW1\ closed}} = -V_{out}|_{\mathrm{SW1\ open}}$. While we were able to get rid of the negative sign, we have an extra R_4 term. Let's look for an extra condition on R_4 to force the relationship we want:

$$V_{out}|_{\mathrm{SW1\ closed}} = -V_{out}|_{\mathrm{SW1\ open}}$$

$$\frac{1}{2C_2}\left(\frac{1}{R_4} - \frac{1}{R_1}\right)\int V_{in}\,\mathrm{dt} = -\left(\frac{-1}{2R_1C_2}\int V_{in}\,\mathrm{dt}\right)$$

$$\frac{1}{2C_2}\left(\frac{1}{R_4} - \frac{1}{R_1}\right)\int V_{in}\,\mathrm{dt} = \frac{1}{2R_1C_2}\int V_{in}\,\mathrm{dt}$$

$$\frac{1}{R_4} - \frac{1}{R_1} = \frac{1}{R_1}$$

$$\boxed{R_4 = \frac{R_1}{2}}$$

As long as R_4 is half the value of R_1, this circuit will act as an ideal integrator.

Conclusion

For design purposes, the fundamental equations for this integrator are

$$V_{out} = \frac{1}{2R_1C_2}\int V_{in}\,\mathrm{dt}$$

$$R_4 = \frac{R_1}{2}$$

Thinking about how an integrator/Schmitt trigger oscillator works, if the integrator integrates faster (R_1C_2 is smaller), the input to the Schmitt trigger will reach it's switching thresholds faster, which will make the frequency increase. This suggests that if you provide the user two capacitors to switch between, you could have both a VCO and voltage controlled LFO.

To be thorough, I redid the analysis assuming $R_2 \neq R_3$. It just resulted in a more complicated gain factor, which wasn't useful:

$$V_{out} = \frac{1}{R_1C_2}\left(\frac{R_3}{R_2 + R_3} + \frac{R_3R_1C_2}{R_2 + R_3} - 1\right)\int V_{in}\,\mathrm{dt}$$